

AN  $U_{qp}(\mathfrak{u}_2)$  ROTOR MODEL FOR ROTATIONAL BANDS  
OF SUPERDEFORMED NUCLEI

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**Abstract**

A nonrigid rotor model is developed from the two-parameter quantum algebra  $U_{qp}(\mathfrak{u}_2)$ . [This model presents the  $U_{qp}(\mathfrak{u}_2)$  symmetry and shall be referred to as the  $qp$ -rotor model.] A rotational energy formula as well as a  $qp$ -deformation of E2 reduced transition probabilities are derived. The  $qp$ -rotor model is applied (through fitting procedures) to twenty rotational bands of superdeformed nuclei in the  $A \sim 130, 150$  and  $190$  mass regions. Systematic comparisons between the  $qp$ -rotor model and the  $q$ -rotor model of Raychev, Roussev and Smirnov, on one hand, and a basic three-parameter model, on the other hand, are performed on energy spectra, on dynamical moments of inertia and on  $B(E2)$  values. The physical signification of the deformation parameters  $q$  and  $p$  is discussed.

# 1 Introduction

Quantum groups and quantum algebras, introduced at the beginning of the eighties,<sup>1–5</sup> continue to attract much attention both in mathematics and physics. For the Physicist, a quantum algebra is commonly considered as a deformation ( $q$ -deformation) of a given Lie algebra. During the last four years, several works have been performed on two-parameter quantum algebras and quantum groups ( $qp$ -deformations).<sup>6–17</sup> (For an elementary introduction to  $q$ - and  $qp$ -quantum algebras, the reader should consult Ref. 14.)

Most of the physical applications, ranging from chemical physics to particle physics, have been mainly concerned up to now with one-parameter quantum algebras ( $q$ -deformations). In particular, in nuclear physics we may mention applications to rotational spectroscopy of deformed and superdeformed nuclei,<sup>18–26</sup> to the interacting boson model,<sup>27,28</sup> to the Moszkowski model,<sup>29,30</sup> to the  $U(3)$  shell model<sup>31</sup> and to the Lipkin-Meshkov-Glick model.<sup>32</sup> There exist also applications to particle physics, as for example to quote a few, to hadron mass formulas<sup>33,34</sup> and to Veneziano amplitudes.<sup>35,36</sup> Among the just mentioned applications, only the ones in Refs. 25 and 36 rely on the use of two-parameter deformations.

The aim of the present paper is two-fold: (i) to further develop and (ii) to apply (to rotational bands of superdeformed nuclei) the nonrigid rotor model briefly introduced in Ref. 25. The latter model, referred to as the  $qp$ -rotor model, is based on the two-parameter quantum algebra  $U_{qp}(u_2)$  while the  $q$ -rotor models introduced by Iwao<sup>18</sup> and Raychev, Roussev and Smirnov<sup>19</sup> (see also Refs. 20–23 and 26) are based on the one-parameter quantum algebra  $U_q(su_2)$ . One of the objectives of this work is to show what we gain when introducing a second “quantum algebra”-type parameter, i.e., when passing from the  $U_q(su_2)$  symmetry to the  $U_{qp}(u_2)$  symmetry.

The organization of this paper is as follows. The  $qp$ -rotor model is introduced in Sec. 2. Subsection 2.1 deals with the mathematical ingredients of the model. The  $qp$ -rotor model itself is developed in Subsec. 2.2 (rotational energy formula) and in Subsec. 2.3 (E2 transition probabilities). Section 3 is devoted to the application of the  $qp$ -rotor model to the description of superdeformed (SD) bands of nuclei in the  $A \sim 130, 150$  and  $190$  mass regions. The results obtained from the  $qp$ -rotor model for rotational energy spectra, dynamical moments of inertia

and  $B(E2)$  values are compared to the ones derived from the  $q$ -rotor model and from a basic (*à la* Bohr-Mottelson) model. Finally, some concluding remarks are presented in Sec. 4.

## 2 A $qp$ -Rotor Model

### 2.1 The quantum algebra $U_{qp}(\mathfrak{u}_2)$

The quantum algebra  $U_{qp}(\mathfrak{u}_2)$  can be constructed from two pairs, say  $\{\tilde{a}_+^+, \tilde{a}_+\}$  and  $\{\tilde{a}_-^+, \tilde{a}_-\}$ , of  $qp$ -deformed (creation and annihilation) boson operators. The action of these  $qp$ -bosons on a nondeformed two-particle Fock space  $\{|n_+, n_-\rangle : n_+ \in \mathbf{N}, n_- \in \mathbf{N}\}$  is controlled by

$$\begin{aligned}\tilde{a}_+^+ |n_+, n_-\rangle &= \sqrt{[[n_+ + \frac{1}{2} + \frac{1}{2}]_{qp}}} |n_+ + 1, n_-\rangle, \\ \tilde{a}_+ |n_+, n_-\rangle &= \sqrt{[[n_+ + \frac{1}{2} - \frac{1}{2}]_{qp}}} |n_+ - 1, n_-\rangle, \\ \tilde{a}_-^+ |n_+, n_-\rangle &= \sqrt{[[n_- + \frac{1}{2} + \frac{1}{2}]_{qp}}} |n_+, n_- + 1\rangle, \\ \tilde{a}_- |n_+, n_-\rangle &= \sqrt{[[n_- + \frac{1}{2} - \frac{1}{2}]_{qp}}} |n_+, n_- - 1\rangle.\end{aligned}\tag{1}$$

In the present paper, we use the notations

$$[[X]]_{qp} := \frac{q^X - p^X}{q - p}\tag{2}$$

and

$$[X]_q := [[X]]_{qq^{-1}} = \frac{q^X - q^{-X}}{q - q^{-1}},\tag{3}$$

where  $X$  may stand for an operator or a (real) number. For Hermitean conjugation requirements, the values of the parameters  $q$  and  $p$  must be restricted to some domains that can be classified as follows: (i)  $q \in \mathbf{R}$  and  $p \in \mathbf{R}$ , (ii)  $q \in \mathbf{C}$  and  $p \in \mathbf{C}$  with  $p = q^*$  (the  $*$  indicates complex conjugation), and (iii)  $q = p^{-1} = e^{i\beta}$  with  $0 \leq \beta < 2\pi$ . The two pairs  $\{\tilde{a}_+^+, \tilde{a}_+\}$  and  $\{\tilde{a}_-^+, \tilde{a}_-\}$  of  $qp$ -bosons commute and satisfy

$$\tilde{a}_\pm \tilde{a}_\pm^+ = [[N_\pm + 1]]_{qp}, \quad \tilde{a}_\pm^+ \tilde{a}_\pm = [[N_\pm]]_{qp},\tag{4}$$

where  $N_+$  and  $N_-$  are the usual number operators with

$$N_{\pm} |n_+, n_-\rangle = n_{\pm} |n_+, n_-\rangle. \quad (5)$$

Of course, the  $qp$ -bosons  $\tilde{a}_{\pm}^{\pm}$  and  $\tilde{a}_{\pm}$  reduce to ordinary bosons (denoted as  $a_{\pm}^{\pm}$  and  $a_{\pm}$  in Refs. 37 and 38 and in Subsec. 2.3) in the limiting situation where  $p = q^{-1} \rightarrow 1$ .

The passage from the (harmonic oscillator) state vectors  $|n_+, n_-\rangle$  to angular momentum state vectors  $|I, M\rangle$  is achieved through the relations

$$I := \frac{1}{2}(n_+ + n_-), \quad M := \frac{1}{2}(n_+ - n_-) \quad (6)$$

and

$$|I, M\rangle \equiv |I + M, I - M\rangle = |n_+, n_-\rangle. \quad (7)$$

Equations (1) may thus be rewritten as

$$\begin{aligned} \tilde{a}_{\pm}^{\pm} |I, M\rangle &= \sqrt{[[I \pm M + \frac{1}{2} + \frac{1}{2}]_{qp}]} |I + \frac{1}{2}, M \pm \frac{1}{2}\rangle, \\ \tilde{a}_{\pm} |I, M\rangle &= \sqrt{[[I \pm M + \frac{1}{2} - \frac{1}{2}]_{qp}]} |I - \frac{1}{2}, M \mp \frac{1}{2}\rangle, \end{aligned} \quad (8)$$

so that the  $qp$ -bosons behave as ladder operators for the quantum numbers  $I$  and  $M$  (with  $|M| \leq I$ ).

We are now in a position to introduce a  $qp$ -deformation of the Lie algebra  $\mathfrak{u}_2$ . A simple calculation shows that the four operators  $J_{\alpha}$  ( $\alpha = 0, 3, +, -$ ) given by

$$J_0 := \frac{1}{2}(N_+ + N_-), \quad J_3 := \frac{1}{2}(N_+ - N_-), \quad J_+ := \tilde{a}_+^{\pm} \tilde{a}_-, \quad J_- := \tilde{a}_-^{\pm} \tilde{a}_+ \quad (9)$$

satisfy the following commutation relations<sup>14,39</sup>

$$[J_3, J_{\pm}] = \pm J_{\pm}, \quad [J_+, J_-] = (qp)^{J_0 - J_3} [[2J_3]]_{qp}, \quad [J_0, J_{\alpha}] = 0. \quad (10)$$

We refer to  $U_{qp}(\mathfrak{u}_2)$  the (quantum) algebra described by (10). To endow  $U_{qp}(\mathfrak{u}_2)$  with a Hopf algebraic structure, it is necessary to introduce a co-product  $\Delta_{qp}$ . The latter co-product is such that:<sup>14</sup>

$$\begin{aligned} \Delta_{qp}(J_0) &= J_0 \otimes 1 + 1 \otimes J_0, \\ \Delta_{qp}(J_3) &= J_3 \otimes 1 + 1 \otimes J_3, \\ \Delta_{qp}(J_{\pm}) &= J_{\pm} \otimes (qp)^{\frac{1}{2}J_0} (qp^{-1})^{+\frac{1}{2}J_3} + (qp)^{\frac{1}{2}J_0} (qp^{-1})^{-\frac{1}{2}J_3} \otimes J_{\pm} \end{aligned} \quad (11)$$

and is clearly seen to depend on the two parameters  $q$  and  $p$ . [Note that with the constraint  $p = q^*$ , to be used in Subsec. 2.2, the co-product satisfies the Hermitean conjugation property  $(\Delta_{qp}(J_\pm))^\dagger = \Delta_{pq}(J_\mp)$  and is compatible with the commutation relations for the four operators  $\Delta_{qp}(J_\alpha)$  (with  $\alpha = 0, 3, +, -$ ).] The universal  $\mathcal{R}$ -matrix (for the coupling of two angular momenta  $I = \frac{1}{2}$ ) associated to the co-product  $\Delta_{qp}$  reads

$$\mathcal{R}_{pq} = \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & \sqrt{pq} & 0 & 0 \\ 0 & p - q & \sqrt{pq} & 0 \\ 0 & 0 & 0 & p \end{pmatrix}, \quad (12)$$

and it can be proved that  $\mathcal{R}_{pq}$  verifies the so-called Yang-Baxter equation.

The operator defined by<sup>14</sup>

$$C_2(U_{qp}(\mathfrak{u}_2)) := \frac{1}{2}(J_+J_- + J_-J_+) + \frac{1}{2}[[2]]_{qp}(qp)^{J_0-J_3}([J_3]_{qp})^2 \quad (13)$$

is an invariant of the quantum algebra  $U_{qp}(\mathfrak{u}_2)$ . It depends truly on the two parameters  $q$  and  $p$ . The invariant  $C_2(U_{qp}(\mathfrak{u}_2))$  will be one of the main mathematical ingredients for the  $qp$ -rotor model to be developed below. Hence, it is worth to examine its structure more precisely, especially its dependence on two independent parameters. Equation (11) suggests the following change of parameters

$$Q := (qp^{-1})^{\frac{1}{2}}, \quad P := (qp)^{\frac{1}{2}}. \quad (14)$$

Then, by introducing the generators  $A_\alpha$  ( $\alpha = 0, 3, +, -$ )

$$A_0 := J_0, \quad A_3 := J_3, \quad A_\pm := (qp)^{-\frac{1}{2}(J_0 - \frac{1}{2})} J_\pm, \quad (15)$$

it can be shown that the two-parameter quantum algebra  $U_{qp}(\mathfrak{u}_2)$  is isomorphic to the central extension

$$U_{qp}(\mathfrak{u}_2) = \mathfrak{u}_1 \otimes U_Q(\mathfrak{su}_2), \quad (16)$$

where  $\mathfrak{u}_1$  is spanned by the operator  $A_0$  and  $U_Q(\mathfrak{su}_2)$  by the set  $\{A_3, A_+, A_-\}$ . The  $Q$ -deformation  $U_Q(\mathfrak{su}_2)$  (a one-parameter deformation!) of the Lie algebra  $\mathfrak{su}_2$  corresponds to the usual commutation relations

$$[A_3, A_\pm] = \pm A_\pm, \quad [A_+, A_-] = [2A_3]_Q. \quad (17)$$

Furthermore, the co-product relations (11) leads to

$$\Delta_{qp}(J_{\pm}) = P^{\Delta_Q(A_0)-\frac{1}{2}} \Delta_Q(A_{\pm}), \quad (18)$$

where the co-product  $\Delta_Q$  is given via

$$\begin{aligned} \Delta_Q(A_0) &= A_0 \otimes 1 + 1 \otimes A_0, \\ \Delta_Q(A_3) &= A_3 \otimes 1 + 1 \otimes A_3, \\ \Delta_Q(A_{\pm}) &= A_{\pm} \otimes Q^{+A_3} + Q^{-A_3} \otimes A_{\pm}. \end{aligned} \quad (19)$$

Equations (17) involve only one parameter, i.e., the parameter  $Q$ . However, two parameters ( $Q$  and  $P$ ) occur in (18) as well as in the invariant  $C_2(U_{qp}(\mathfrak{u}_2))$  transcribed in terms of  $Q$  and  $P$ . As a matter of fact, (13) can be rewritten as

$$C_2(U_{qp}(\mathfrak{u}_2)) = P^{2A_0-1} C_2(U_Q(\mathfrak{su}_2)), \quad (20)$$

where

$$C_2(U_Q(\mathfrak{su}_2)) := \frac{1}{2} (A_+ A_- + A_- A_+) + \frac{1}{2} [2]_Q ([A_3]_Q)^2 \quad (21)$$

is an invariant of  $U_Q(\mathfrak{su}_2)$  [compare Eqs. (13) and (21)]. As a consequence, of central importance for the  $qp$ -rotor model of Subsec. 2.2, the invariant  $C_2(U_{qp}(\mathfrak{u}_2))$ , in either the form (13) or the form (20), depends on two parameters. Finally, it should be noted that  $C_2(U_{qp}(\mathfrak{u}_2))$  can be identified to the invariant of  $U_q(\mathfrak{su}_2)$  and to the Casimir of  $\mathfrak{su}_2$  when  $p = q^{-1}$  and  $p = q^{-1} \rightarrow 1$ , respectively. In this sense, the  $U_{qp}(\mathfrak{u}_2)$  symmetry encompasses the  $U_q(\mathfrak{su}_2)$  and  $\mathfrak{su}_2$  symmetries.

To close this section, let us say a few words on the representation theory of  $U_{qp}(\mathfrak{u}_2)$  in the case where neither  $q$  nor  $p$  are roots of unity. An irreducible representation of this quantum algebra is described by a Young pattern  $[\varphi_1; \varphi_2]$  with  $\varphi_1 - \varphi_2 = 2I$ , where  $2I$  is a nonnegative integer ( $I$  will represent a spin angular momentum in what follows). We note  $[[\varphi_1; \varphi_2], M)$  (with  $M = -I, -I+1, \dots, +I$ ) the basis vectors for the representation  $[\varphi_1; \varphi_2]$ .

We are now ready to develop a  $qp$ -rotor model for describing energy levels and transition probabilities for deformed and superdeformed nuclei.

## 2.2 Energy levels

We want to construct a nonrigid rotor model. As a first basic hypothesis (Hypothesis 1), we take a rigid rotor with  $U_{qp}(u_2)$  symmetry, thus introducing the nonrigidity through the  $qp$ -deformation of the Lie algebra  $u_2$ . More precisely, we assume that the  $qp$ -rotor Hamiltonian  $H$  is a linear function of the invariant  $C_2(U_{qp}(u_2))$ :

$$H = \frac{1}{2\mathcal{I}} C_2(U_{qp}(u_2)) + E_0, \quad (22)$$

where  $\mathcal{I}$  denotes the moment of inertia of the rotor and  $E_0$  the bandhead energy. As a second hypothesis (Hypothesis 2), we take  $\varphi_1 = 2I$  and  $\varphi_2 = 0$ . This means that we work with state vectors of the type  $|I, M) \equiv |[2I; 0], M)$ . Therefore, the eigenvalues of  $H$  are obtained by the action of  $H$  on the physical subspace  $\{|I, M) : M = -I, -I + 1, \dots, +I\}$  of constant angular momentum  $I$  corresponding to the irreducible representation  $[2I; 0]$  of  $U_{qp}(u_2)$ . The two preceding hypotheses lead to the energy formula

$$E(I)_{qp} = \frac{1}{2\mathcal{I}} [[I]]_{qp} [[I + 1]]_{qp} + E_0 \quad (23)$$

for the  $qp$ -deformed rotational level of angular momentum  $I$ .

By introducing  $s = \ln q$  and  $r = \ln p$ , Eq. (23) yields

$$E(I)_{qp} = \frac{1}{2\mathcal{I}} e^{(2I-1)\frac{s+r}{2}} \frac{\sinh(I\frac{s-r}{2}) \sinh[(I+1)\frac{s-r}{2}]}{\sinh^2(\frac{s-r}{2})} + E_0. \quad (24)$$

Preliminary studies have lead us to the conclusion that a good agreement between theory and experiment cannot be always obtained by varying the parameters  $s$  and  $r$  (or  $q$  and  $p$ ) on the real line  $\mathbf{R}$ , a fact that confirms a similar conclusion reached in Ref. 20 for  $p = q^{-1} \in \mathbf{R}$ . In addition, if we want that our  $qp$ -rotor model reduces to the  $q$ -rotor model developed by Raychev, Roussev and Smirnov<sup>19</sup> when  $p = q^{-1}$  (or equivalently  $r = -s$ ), we are naturally left to impose that  $(s+r)$  and  $(s-r)/i$  should be real numbers. [Observe that the two constraints  $(s+r) \in \mathbf{R}$  and  $(s-r)/i \in \mathbf{R}$  ensure that the energy  $E(I)_{qp}$  is real as it should be.] Furthermore, we shall see that for certain SD bands, a good agreement between theory and experiment requires that the parameters  $s$  and  $r$  vary on the real line  $\mathbf{R}$ . Thus, we shall consider the two possible

parametrizations:

$$\begin{aligned}
\text{(a)} \quad \frac{s+r}{2} &= \beta \cos \gamma \in \mathbf{R}, & \frac{s-r}{2i} &= \beta \sin \gamma \in \mathbf{R}, \\
\text{(b)} \quad \frac{s+r}{2} &= \beta \cos \gamma \in \mathbf{R}, & \frac{s-r}{2i} &= \frac{\beta \sin \gamma}{i} \in i\mathbf{R},
\end{aligned} \tag{25}$$

so that the parameters  $q$  and  $p$  read

$$\begin{aligned}
\text{(a)} \quad q &= e^{\beta \cos \gamma} e^{+i\beta \sin \gamma}, & p = q^* &= e^{\beta \cos \gamma} e^{-i\beta \sin \gamma}, \\
\text{(b)} \quad q &= e^{\beta \cos \gamma} e^{+\beta \sin \gamma}, & p &= e^{\beta \cos \gamma} e^{-\beta \sin \gamma}.
\end{aligned} \tag{26}$$

Thus, our  $qp$ -rotor model involves two independent real parameters  $\beta$  and  $\gamma$  corresponding either to (a) the two complex parameters  $q$  and  $p$  subjected to the constraint  $p = q^*$  or to (b) the two real parameters  $q$  and  $p$ . In terms of the parameters  $\beta$  and  $\gamma$ , the energy formula (24) takes the form

$$E(I)_{qq^*} = \frac{1}{2\mathcal{I}} e^{(2I-1)\beta \cos \gamma} \frac{\sin(I\beta \sin \gamma) \sin[(I+1)\beta \sin \gamma]}{\sin^2(\beta \sin \gamma)} + E_0 \tag{27a}$$

or

$$E(I)_{qp} = \frac{1}{2\mathcal{I}} e^{(2I-1)\beta \cos \gamma} \frac{\sinh(I\beta \sin \gamma) \sinh[(I+1)\beta \sin \gamma]}{\sinh^2(\beta \sin \gamma)} + E_0 \tag{27b}$$

in the parametrizations of type (a) or (b), respectively. We shall use both Eqs. (27a) and (27b) in our fitting procedures.

In the (a)-parametrization, to better understand the connection between our  $qp$ -rotor model and the  $q$ -rotor model of Ref. 19, we can perform a series analysis of Eq. (27a). A straightforward calculation allows us to rewrite Eq. (27a) as

$$E(I)_{qq^*} = \frac{1}{2\mathcal{I}_{\beta\gamma}} \left( \sum_{n=0}^{\infty} d_n(\beta, \gamma) [I(I+1)]^n + (2I+1) \sum_{n=0}^{\infty} c_n(\beta, \gamma) [I(I+1)]^n \right) + E_0, \tag{28}$$

where the expansion coefficients  $d_n(\beta, \gamma)$  and  $c_n(\beta, \gamma)$  are given in turn by the series

$$d_n(\beta, \gamma) = \frac{2^{2n-1}}{\sin^2(\beta \sin \gamma)}$$



$$\begin{aligned}
& \times \sum_{k=0}^{\infty} \left\{ (\cos \gamma)^{2k+2n} \cos(\beta \sin \gamma) - \cos[(2k+2n)\gamma] \right\} \frac{\beta^{2k+2n}}{(2k+2n)!} \frac{(k+n)!}{k! n!}, \\
c_n(\beta, \gamma) &= \frac{2^{2n-1}}{\sin^2(\beta \sin \gamma)} \\
& \times \sum_{k=0}^{\infty} \left\{ (\cos \gamma)^{2k+2n+1} \cos(\beta \sin \gamma) - \cos[(2k+2n+1)\gamma] \right\} \frac{\beta^{2k+2n+1}}{(2k+2n+1)!} \frac{(k+n)!}{k! n!}. \quad (29)
\end{aligned}$$

In Eq. (28), we have introduced the deformed moment of inertia:

$$\mathcal{I}_{\beta\gamma} = \mathcal{I} e^{2\beta \cos \gamma}, \quad (30)$$

which gives back the ordinary moment of inertia when  $\gamma = \pi/2$  (i.e.,  $q = p^{-1} = e^{i\beta}$ ). In the limiting situation where  $\gamma = \pi/2$ , the coefficients  $c_n(\beta, \gamma)$  vanish and the energy formula (28) simplifies to

$$E(I)_{qq^{-1}} = \frac{1}{2\mathcal{I}} \frac{\beta^2}{\sin^2 \beta} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^{n-1}}{n!} \beta^{n-1} j_{n-1}(\beta) [I(I+1)]^n + E_0, \quad (31)$$

where  $j_{n-1}$  denotes a spherical Bessel function of the first kind. Equation (31) was derived by Bonatsos *et al.*<sup>20</sup> for the  $q$ -rotor model with  $q = e^{i\beta}$  in order to prove the mathematical parentage between the  $q$ -rotor model and the variable moment of inertia (VMI) model.<sup>40–42</sup> The series (31) corresponds indeed to the compact expression

$$E(I)_{qq^{-1}} = \frac{1}{2\mathcal{I}} [I]_q [I+1]_q + E_0, \quad (32)$$

to be compared with Eq. (23). Note that Eq. (32) corresponds also to the (b)-parametrization with  $\gamma = \pi/2$ . The transition from Eq. (23) to Eq. (32) illustrates the descent from the  $U_{qp}(\mathfrak{u}_2)$  symmetry of the  $qp$ -rotor to the  $U_q(\mathfrak{su}_2)$  symmetry of the  $q$ -rotor. A further descent in symmetry is obtained when  $\beta \rightarrow 0$  (i.e.,  $q = p^{-1} \rightarrow 1$ ): in this case  $[I]_q [I+1]_q \rightarrow I(I+1)$  and we get [from Eq. (32)] the usual energy formula corresponding to the rigid rotor with  $\mathfrak{su}_2$  symmetry.

## 2.3 E2 transition probabilities

We now examine the implication of the  $U_{qp}(\mathfrak{u}_2)$  symmetry on the calculation of the electric quadrupole transition probability  $T(E2; I+2 \rightarrow I)$ . Let us start with the ordinary expression

of the reduced transition probability, namely,

$$B(\text{E2}; I+2 \rightarrow I) = \frac{5}{16\pi} Q_0^2 \left| (I+2, M, 2, 0 | I+2, 2, I, M) \right|^2 \quad (33)$$

for an E2 transition.<sup>43</sup> In Eq. (33),  $Q_0$  is the intrinsic electric quadrupole moment in the body-fixed frame. The coefficient of type  $(j, m, k, \mu | j, k, j', m')$  in the right-hand side of Eq. (33) is a usual Clebsch-Gordan coefficient for the group  $\text{SU}(2)$ . Our goal is to find a  $qp$ -analog of  $T(\text{E2}; I+2 \rightarrow I)$  and, thus, of Eq. (33). The strategy for obtaining a  $qp$ -analog of  $B(\text{E2}; I+2 \rightarrow I)$  is the following:

(i) We first rewrite the  $\text{SU}(2)$  Clebsch-Gordan coefficient of Eq. (33) in terms of a matrix element of an  $\text{SU}(2)$  unit tensor operator  $t_{k\mu\alpha}$  with  $k=2$ ,  $\mu=0$  and  $\alpha=-2$ . This may be done from the general formula<sup>38</sup>

$$(j', m' | t_{k\mu\alpha} | j, m) = \delta(j', j+\alpha) \delta(m', m+\mu) (-1)^{2k} (2j'+1)^{-\frac{1}{2}} (j, m, k, \mu | j, k, j', m'), \quad (34)$$

which shows that the irreducible tensor operator  $t_{k\mu\alpha}$  produces the (angular momentum) state vector  $|j+\alpha, m+\mu\rangle$  when acting upon the state vector  $|j, m\rangle$ . Then, Eq. (33) is amenable to the form

$$B(\text{E2}; I+2 \rightarrow I) = \frac{5}{16\pi} Q_0^2 (2I+1) \left| (I, M | t_{20-2} | I+2, M) \right|^2 \quad (35)$$

by making use of Eq. (34).

(ii) We know that the general operator  $t_{k\mu\alpha}$  can be realized in terms of two pairs  $\{a_+^\pm, a_+^\pm\}$  and  $\{a_-^\pm, a_-^\pm\}$  of ordinary boson operators. In this respect, we may consider the so-called van der Waerden<sup>38</sup> realization of  $t_{k\mu\alpha}$ . There are several ways to  $qp$ -deform the operator  $t_{k\mu\alpha}$ . Here, we choose to define a  $qp$ -deformation  $t_{k\mu\alpha}(qp)$  by replacing, in the van der Waerden realization of  $t_{k\mu\alpha}$ , the ordinary bosons  $\{a_\pm^\pm, a_\pm^\pm\}$  by  $qp$ -deformed bosons  $\{\tilde{a}_\pm^\pm, \tilde{a}_\pm^\pm\}$  and the ordinary factorials  $x!$  by  $qp$ -deformed factorials  $[[x]]_{qp}! = [[x]]_{qp} [[x-1]]_{qp} \cdots [[1]]_{qp}$  for  $x \in \mathbf{N}$ . We thus obtain

$$\begin{aligned} t_{k\mu\alpha}(qp) &= (-1)^{k+\alpha} \left( \frac{[[k+\mu]]_{qp}! [[k-\mu]]_{qp}! [[k+\alpha]]_{qp}! [[k-\alpha]]_{qp}! [[2j-k+\alpha]]_{qp}!}{[[2j+k+\alpha+1]]_{qp}!} \right)^{\frac{1}{2}} \\ &\times \sum_z (-1)^z \frac{(\tilde{a}_-^+)^{k-\mu-z} (\tilde{a}_-^-)^{k-\alpha-z} (\tilde{a}_+^+)^{\mu+\alpha+z} (\tilde{a}_+^-)^z}{[[k-\mu-z]]_{qp}! [[k-\alpha-z]]_{qp}! [[\mu+\alpha+z]]_{qp}! [[z]]_{qp}!}. \end{aligned} \quad (36)$$

In particular, the  $qp$ -deformed operator  $t_{20-2}(qp)$  connecting the state vector  $|I + 2, M\rangle$ , with  $j \equiv I + 2$ , to the state vector  $|I, M\rangle$ , with  $j' \equiv I$ , reads

$$t_{20-2}(qp) = \left( \frac{[[3]]_{qp} [[4]]_{qp} [[2I]]_{qp}!}{[[2]]_{qp} [[2I + 5]]_{qp}!} \right)^{\frac{1}{2}} (\tilde{a}_+)^2 (\tilde{a}_-)^2, \quad (37)$$

an expression of direct interest for deriving the  $qp$ -analog of  $B(E2; I + 2 \rightarrow I)$ .

(iii) We assume that the  $qp$ -analog  $B(E2; I + 2 \rightarrow I)_{qp}$  of  $B(E2; I + 2 \rightarrow I)$  is simply

$$B(E2; I + 2 \rightarrow I)_{qp} := \frac{5}{16\pi} Q_0^2 [[2I + 1]]_{qp} \left| (I, M | t_{20-2}(qp) | I + 2, M) \right|^2. \quad (38)$$

[Equation (38) constitutes the third and last hypothesis (Hypothesis 3) for our  $qp$ -rotor model.] By using Eqs. (37) and (8), the relevant matrix element of the operator  $t_{20-2}(qp)$  is easily found to be

$$(I, M | t_{20-2}(qp) | I + 2, M) = \left( \frac{[[3]]_{qp} [[4]]_{qp} [[2I]]_{qp}! [[I + M + 1]]_{qp} [[I - M + 1]]_{qp} [[I + M + 2]]_{qp} [[I - M + 2]]_{qp}}{[[2]]_{qp} [[2I + 5]]_{qp}!} \right)^{\frac{1}{2}}. \quad (39)$$

Then, the introduction of Eq. (39) into Eq. (38) yields

$$B(E2; I + 2 \rightarrow I)_{qp} = \frac{5}{16\pi} Q_0^2 \frac{[[3]]_{qp} [[4]]_{qp}}{[[2]]_{qp}} \frac{([ [I + 1] ]_{qp} [ [I + 2] ]_{qp})^2}{[[2I + 2]]_{qp} [[2I + 3]]_{qp} [[2I + 4]]_{qp} [[2I + 5]]_{qp}} \quad (40)$$

in the case of the  $K \equiv M = 0$  bands.

For the purpose of comparison with experimental results, we must calculate the E2 transition probability  $T(E2; I + 2 \rightarrow I)$  in the  $qp$ -deformed scheme. We define such a probability by

$$T(E2; I + 2 \rightarrow I)_{qp} := 1.223 \cdot 10^9 (E_\gamma(I + 2)_{qp})^5 B(E2; I + 2 \rightarrow I)_{qp}. \quad (41)$$

Equation (41) turns out to be a simple  $qp$ -deformation of the usual E2 transition probability. [In Eq. (41),  $T(E2; I + 2 \rightarrow I)_{qp}$  is in units of  $\text{sec}^{-1}$ ,  $E_\gamma(I + 2)_{qp} := E(I + 2)_{qp} - E(I)_{qp}$  in units of MeV, and  $B(E2; I + 2 \rightarrow I)_{qp}$  in units of  $\text{e}^2\text{fm}^4$ .]

At this stage, a contact with the formula  $B(E2; I + 2 \rightarrow I)_q$  derived by Raychev, Roussev and Smirnov<sup>19</sup> is in order. First, by taking  $p = q^{-1}$  the right-hand side of (40) may be specialized

to the expression of  $B(\text{E}2; I + 2 \rightarrow I)_q$  obtained in Ref. 19. Hence, our  $qp$ -rotor model for the E2 transition probability admits as a particular case the corresponding  $q$ -rotor model worked out in Ref. 19. Second, it can be shown that

$$B(\text{E}2; I + 2 \rightarrow I)_{qp} = P^{-4(I+1)} B(\text{E}2; I + 2 \rightarrow I)_Q, \quad (42)$$

where  $P$  and  $Q$  are given by (14).

Let us close with a remark. Should we have chosen to find a  $qp$ -analog of the Clebsch-Gordan coefficient in (33), we would have obtained<sup>39</sup>

$$(I + 2, M, 2, 0 | I + 2, 2, I, M)_{qp} = (I + 2, M, 2, 0 | I + 2, 2, I, M)_Q \quad (43)$$

and, consequently

$$B(\text{E}2; I + 2 \rightarrow I)_{qp} = B(\text{E}2; I + 2 \rightarrow I)_Q. \quad (44)$$

We prefer to use (42) rather than (44) because the factorization in (42) parallels the one in (20).

## 3 Description of Superdeformed Bands

### 3.1 Fitting procedure

The  $qp$ -rotor model developed in Sec. 2 was applied to twenty rotational SD bands of nuclei in the  $A \sim 130, 150$  and  $190$  mass regions. The  $\gamma$ -ray energies

$$E_\gamma(I) := E(I) - E(I - 2) \quad (45)$$

were computed from the energy formula

$$E(I) \equiv E(I)_{qq^*} = \frac{1}{2\mathcal{I}} e^{(2I-1)a} \frac{\sin(Ib) \sin[(I+1)b]}{\sin^2(b)} + E_0 \quad (46a)$$

or

$$E(I) \equiv E(I)_{qp} = \frac{1}{2\mathcal{I}} e^{(2I-1)a} \frac{\sinh(Ib) \sinh[(I+1)b]}{\sinh^2(b)} + E_0 \quad (46b)$$

that correspond to Eq. (27a) or (27b), respectively, with

$$a := \beta \cos \gamma, \quad b := \beta \sin \gamma. \quad (47)$$

The free parameters of the  $qp$ -rotor model are then  $a$ ,  $b$  and  $\mathcal{I}$ .

For the sake of comparison, we also computed the transition energies  $E_\gamma(I)$  and performed an analysis of the same SD bands from two other models. First, we used

$$E(I) \equiv E(I)_{qq^{-1}} = \frac{1}{2\mathcal{I}'} \frac{\sin(I\beta') \sin[(I+1)\beta']}{\sin^2 \beta'} + E_0, \quad (48)$$

where  $\beta'$  (defined by  $q = e^{i\beta'}$ ) and  $\mathcal{I}'$  are the two free parameters of the  $q$ -rotor model of Ref. 19. Second, we also applied the energy formula

$$E(I) \equiv A'I(I+1) + B'[I(I+1)]^2 + C'[I(I+1)]^3 + E_0 \quad (49)$$

arising from the (Bohr-Mottelson) basic model<sup>43</sup> restricted to three free parameters, i.e.,  $A'$ ,  $B'$  and  $C'$ .

For each of the three models, the parameters ( $a$ ,  $b$  and  $\mathcal{I}$  for the  $qp$ -rotor;  $\beta'$  and  $\mathcal{I}'$  for the  $q$ -rotor;  $A'$ ,  $B'$  and  $C'$  for the basic model) were fixed by minimizing

$$\chi := \sqrt{\frac{1}{n-m} \sum_I \left[ \frac{E_\gamma^{\text{th}}(I) - E_\gamma^{\text{ex}}(I)}{\Delta E_\gamma(I)} \right]^2}, \quad (50)$$

where  $n$  is the number of experimental points included in the fitting procedure,  $m$  is the number of freely varied parameters, and  $\Delta E_\gamma(I)$  are the experimental errors.

## 3.2 Results and discussions

### 3.2.1 Fitting of data

We present in Table 1 the free parameters and the  $\chi$ -values, for the  $q$ - and the  $qp$ -rotor models, obtained from the twenty fitted SD bands. (The tables and figures of this paper can be obtained from the authors.) For space saving purposes, we do not report the corresponding results obtained with the basic model since the  $\chi$ -values are generally higher than the ones derived from the  $qp$ -rotor model (a fact to be confirmed in Subsec. 3.2.2).

Table 1 exhibits two general trends: first, the best results are obtained in the  $A \sim 190$  mass region for the two models and, second, the  $\chi$ -values for the  $qp$ -rotor are better than those for the  $q$ -rotor. Indeed, the  $\chi$ -values obtained for the  $qp$ -rotor (respectively,  $q$ -rotor) are between

0.6 and 4.2 (respectively, 0.9 and 29.4) in the  $A \sim 190$  mass region except for  $^{194}\text{Hg(a)}$  with  $\chi = 9.1$  (respectively, 35.2) while the  $\chi$ -values are between 1.9 and 20.3 (respectively, 8.8 and 87.9) in the two other mass regions. The high values obtained for  $\chi$  are not surprising: in the standard definition of  $\chi$ , Eq. (50), the difference between the theoretical and experimental transition energies is divided by the experimental error  $\Delta E_\gamma(I)$  that is equal to 0.5 keV except for the recent experimental data<sup>44,45</sup> on  $^{192}\text{Hg}$ ,  $^{194}\text{Hg(a)}$  and  $^{194}\text{Hg(b)}$  for which  $\Delta E_\gamma(I)$  is as below as 0.1 keV. Therefore, we may emphasize the excellent quality of the fits for the  $A \sim 190$  bands, especially in the case of the  $qp$ -rotor model.

For the  $qp$ -rotor model, the quality of the fits is connected to the nature (real or complex) of the parameters  $q$  and  $p$ . The best fits were obtained by taking: (i) the (a)-parametrization (i.e.,  $q = e^{a+ib}$  and  $p = e^{a-ib}$ ) for  $^{146}\text{Gd}$  and the 190 SD bands and (ii) the (b)-parametrization (i.e.,  $q = e^{a+b}$  and  $p = e^{a-b}$ ) for the 130 and 150 SD bands. To illustrate our results, we globally characterize in Fig. 1 the twenty SD bands by their position in the plane of the two “quantum algebra”-type real parameters  $a$  and  $b$ . As it was shown in Ref. 20, the parameter  $\beta'$  of the  $q$ -rotor (which occurs in a sine like the parameter  $b$  in the (a)-parametrization of the  $qp$ -rotor) can be interpreted as a softness or stretching parameter of the nucleus, similar to the parameter  $\sigma$  of the VMI model.<sup>40–42</sup> We adopt this interpretation for the parameter  $b$  (that coincides with the parameter  $\beta'$  when  $a = 0$ ) in the (a)-parametrization. Then, the (b)-parametrization describes a distortion phenomenon (decrease of the dynamical moment of inertia with the spin of the nucleus) rather than a stretching phenomenon. In the (a)- and (b)-parametrizations, the role played by the parameter  $a$ , appearing in the exponential term of Eq. (46), is clear. The parameter  $a$  has a crucial effect of correction on the distortion (stretching or anti-stretching) function of the parameter  $b$ . In addition, we note from Table 1 that the sign of  $a$  is the same as that of the difference  $\mathcal{I} - \mathcal{I}'$ . In other words, at high angular momenta, the exponential term in (46) moderates (when  $a < 0$ ) or accentuates (when  $a > 0$ ) the contribution to the energy of  $\frac{1}{2\mathcal{I}}$  with respect to the contribution of  $\frac{1}{2\mathcal{I}'}$ . Therefore, the parameter  $a$  can moderate or accentuate the distortion phenomenon of the nucleus.

Before performing a systematic comparison between the three models under consideration, we present in Tables 2-6 the calculated and experimental transition energies for the  $qp$ -rotor

model. The numerical results in Tables 2-6 confirm the preceding interpretation of the free parameters of the  $qp$ -rotor model. Here again, we note that the quality of the fits is better in the  $A \sim 190$  region than in the  $A \sim 130$  and  $150$  regions. This reflects the fact that the  $\gamma$ -ray energies range from 200 to 900 keV (respectively, 600 to 1700 keV) for angular momenta ranging from 8 to 50 (respectively, 14 to 64) for  $A \sim 190$  (respectively,  $A \sim 130$  and  $150$ ).

### 3.2.2 Comparative analyses

In order to confirm the difference (already evocated in the  $\chi$ -values analysis) between the  $qp$ -rotor model and the basic model, we consider three representative nuclei for each of the considered mass regions. Figure 2 shows the differences between the calculated and experimental transition energies for the nuclei  $^{132}\text{Ce}$ ,  $^{152}\text{Dy}$  and  $^{192}\text{Hg}$  obtained from the basic and  $qp$ -rotor models. It is clear that the  $qp$ -rotor model is more appropriate, in particular for  $^{132}\text{Ce}$ , for describing the distortion phenomenon than the basic model. Therefore, we switch to a detailed comparison between the  $q$ - and  $qp$ -rotor models. Figures 3-7 display the results (in terms of differences as in Fig. 2) afforded by the  $q$ - and  $qp$ -rotor models for the twenty SD bands under study. Two remarks arise from Figs. 3-7. First, the preceding  $\chi$ -values analysis is clearly confirmed. Second, we observe that the  $qp$ -rotor model is much better than the  $q$ -rotor one when the distortion phenomenon is particularly pronounced. For example, in the case of the  $^{192}\text{Hg}$  band that presents nineteen transitions and where the variation of the stretching effect becomes less important at high spin, the  $qp$ -rotor model provides the best results.

An alternative way to analyse the stretching phenomenon in the  $A \sim 190$  region amounts to compare the *theoretical* and *experimental* dynamical moments of inertia

$$\mathcal{I}_{\text{th}}^{(2)}(I) := \left( \frac{d^2 E}{dx^2} \right)^{-1}, \quad E \equiv E(I), \quad x \equiv x(I) := \sqrt{I(I+1)} \quad (51)$$

and

$$\mathcal{I}_{\text{ex}}^{(2)}(I) := \frac{4000}{E_{\gamma}(I+2) - E_{\gamma}(I)}, \quad (52)$$

respectively. The experimental  $\gamma$ -ray energies  $E_{\gamma}$  in (52) are defined by (45) and we take the theoretical energies  $E$  in (51) as given by (46) (respectively, (48)) for the  $qp$ -rotor (respectively,  $q$ -rotor) model. [The dynamical moments in (51) and 52) are in units of  $\hbar^2 \text{MeV}^{-1}$ .] Figure 8

shows the results for four SD bands of the three nuclei  $^{190-192-194}\text{Hg}$ : the experimental moments of inertia are calculated from Refs. 44, 45 and 54 and the theoretical ones by using the free parameters  $\frac{1}{2I}$ ,  $a$ ,  $b$  and  $\frac{1}{2I}$ ,  $\beta'$  for the  $U_{qp}(\text{u}_2)$  and  $U_q(\text{su}_2)$  symmetries, respectively. The  $qp$ -rotor results are much closer to the experimental results than the  $q$ -rotor ones, due to the influence of the parameter  $a$ . [In passing, Fig. 8 shows that globally, both for the  $q$ - and  $qp$ -rotor models, the second derivative of the energy is significative when calculated with the fitted values of the free parameters.]

A last way to compare the  $qp$ -rotor model with the two others is to use experimental values of E2 transition probabilities. From such values, we can compute two different intrinsic electric quadrupole moments, namely,  $(Q_0)_{qp}$  and  $(Q_0)_q$  for the  $U_{qp}(\text{u}_2)$  and  $U_q(\text{su}_2)$  symmetries, respectively. For the  $U_{qp}(\text{u}_2)$  symmetry,  $(Q_0)_{qp}$  is deduced from (40) and (41), where we take the experimental value for the E2 transition probability and all the other terms (including the transition energies) are calculated from the parameters of the  $qp$ -rotor model obtained from the optimization of energy. A similar calculation is conducted for  $(Q_0)_q$  corresponding to the  $U_q(\text{su}_2)$  symmetry. The *experimental* intrinsic electric quadrupole moment  $(Q_0)_{\text{ex}}$  corresponds to the  $\text{su}_2$  symmetry: it is calculated from (40) and (41) with  $q = p^{-1} \rightarrow 1$  by taking the experimental E2 transition probability and the experimental  $\gamma$ -ray energy. We present in Tables 7-9 the values of the quadrupole moments  $(Q_0)_{qp}$ ,  $(Q_0)_q$  and  $(Q_0)_{\text{ex}}$ , together with the experimental errors, computed for  $^{192}\text{Hg}$  and  $^{194-196}\text{Pb}$  with the experimental E2 transition probabilities of Refs. 57, 60 and 61. We see that the values of  $(Q_0)_{qp}$  are in better agreement with the experimental quadrupole moments  $(Q_0)_{\text{ex}}$  than the values of  $(Q_0)_q$ . To further compare the symmetries  $U_{qp}(\text{u}_2)$ ,  $U_q(\text{su}_2)$  and  $\text{su}_2$ , it is interesting to calculate the geometrical factor of type

$$G(I) := \frac{16\pi}{5} \frac{B(\text{E2}; I+2 \rightarrow I)}{5Q_0^2} \quad (53)$$

for the three symmetries. Figure 9 displays this factor as a function of the spin  $I$  for the three nuclei  $^{192}\text{Hg}$  and  $^{194-196}\text{Pb}$ . At high spin, the increasing of  $G(I)$  characterises the two “quantum algebra”-type models, while  $G(I)$  reaches a limit value for the  $\text{su}_2$  symmetry. Note that  $G(I)$  increases less strongly (i.e., more linearly) for the  $U_{qp}(\text{u}_2)$  symmetry than for the  $U_q(\text{su}_2)$  symmetry when the parameter  $b$  differs from the parameter  $\beta'$ .



## 4 Conclusions

In this paper, we concentrated on a new nonrigid rotor model (the  $qp$ -rotor model) based on three hypotheses in the framework of an investigation of the two-parameter quantum algebra  $U_{qp}(\mathfrak{u}_2)$ . The two facets of this model consist of a three-parameter energy level formula and a  $qp$ -deformed E2 transition probability formula. As limiting cases, the  $qp$ -rotor model gives back the  $q$ -rotor model<sup>19</sup> (when  $p = q^{-1}$ ) based on the quantum algebra  $U_q(\mathfrak{su}_2)$  and the rigid rotor model (when  $p = q^{-1} \rightarrow 1$ ) based on the Lie algebra  $\mathfrak{su}_2$ .

Twenty rotational bands of superdeformed nuclei in the  $A \sim 130, 150$  and  $190$  mass regions were used to test our  $qp$ -rotor model and to compare it to the  $q$ -rotor model and to a basic (with a three-term polynomial energy formula) model. The main results may be summarized as follows. First, the  $qp$ -rotor model is better than the  $q$ -rotor model and the basic model as far as energy spectra are concerned. Second, the energy fits for the twenty SD bands are in good agreement with experiment both for the  $q$ - and  $qp$ -rotor models. However, a marked difference between the latter two models manifests itself in the energy spectrum and also in the second derivative of the energy (i.e., for the dynamical moment of inertia). Third, in terms of  $B(\text{E}2)$  values the results afforded by the  $U_{qp}(\mathfrak{u}_2)$  symmetry are between those given by the  $U_q(\mathfrak{su}_2)$  symmetry and the  $\mathfrak{su}_2$  symmetry: the  $B(\text{E}2)$  values for the  $qp$ -rotor model increase more or less linearly with spin, a result that does not hold for the  $q$ -rotor model.

As a general conclusion, the  $qp$ -rotor is appropriate for describing the collective phenomenon of distortion occurring in the rotation of the nucleus (increase or decrease of the dynamical moment of inertia with the spin). The net difference between the  $q$ - and  $qp$ -rotor models comes from the “quantum algebra”-type parameter  $a$  that tends to smooth the (spherical or hyperbolic) sine term in the energy and thus accentuates or moderates the distortion phenomenon of the nucleus.

To close this paper, let us mention that Hypothesis 2 (i.e.,  $\varphi_1 = 2I$  and  $\varphi_2 = 0$ ) of our model might be abandoned. This would lead to a *à la* Dunham formulation for describing more complicated rotational spectra of deformed and superdeformed nuclei or rovibrational spectra of diatomic molecules. As a further extension, it would be also interesting to combine our model

with one of Ref. 24 (based on the  $q$ -Poincaré symmetry) in the case of heavy nuclei. Work in these directions is in progress.

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### Table captions:

Table 1. Free parameters for the  $q$ - and  $qp$ -rotor models:  $\beta'$  corresponds to  $q = e^{i\beta'}$ ;  $a = \beta \cos \gamma$  and  $b = \beta \sin \gamma$  correspond: (i) to  $q = e^{a+ib}$  and  $p = e^{a-ib}$  for  $^{146}\text{Gd}$  and the 190 SD bands, and (ii) to  $q = e^{a+b}$  and  $p = e^{a-b}$  for the 130 and 150 SD bands;  $\frac{1}{2T}$  and  $\frac{1}{2T'}$  are in units of  $\hbar^{-2}\text{keV}$ .

Table 2. Theoretical and experimental  $\gamma$ -ray energies and experimental errors for SD bands in the  $A \sim 130$  region. Experimental data are taken from Refs. 46-48.

Table 3. Theoretical and experimental  $\gamma$ -ray energies and experimental errors for SD bands in the  $A \sim 150$  region. Experimental data are taken from Refs. 49-53.

Table 4. Theoretical and experimental  $\gamma$ -ray energies and experimental errors for SD bands in the  $A \sim 190$  region. Experimental data are taken from Refs. 44, 45 and 54.

Table 5. Theoretical and experimental  $\gamma$ -ray energies and experimental errors for SD bands in the  $A \sim 190$  region. Experimental data are taken from Refs. 55-58.

Table 6. Theoretical and experimental  $\gamma$ -ray energies and experimental errors for SD bands in the  $A \sim 190$  region. Experimental data are taken from Ref. 59.

Table 7. Intrinsic electric quadrupole moments for  $^{192}\text{Hg}$  in units of eb. The theoretical moments  $(Q_0)_q$  and  $(Q_0)_{qp}$  are calculated with  $\frac{1}{2T'} = 5.58$ ,  $\beta' = 0.12 \cdot 10^{-1}$  and  $\frac{1}{2T} = 5.91$ ,  $a = -0.15 \cdot 10^{-2}$ ,  $b = 0.47 \cdot 10^{-2}$  for the  $q$ - and  $qp$ -rotor models, respectively. The experimental values  $(Q_0)_{\text{ex}}$  as well as the upper and lower experimental errors  $\Delta Q_0^+$  and  $\Delta Q_0^-$  are taken from Ref. 60.

Table 8. Intrinsic electric quadrupole moments for  $^{194}\text{Pb}$  in units of eb. The theoretical moments  $(Q_0)_q$  and  $(Q_0)_{qp}$  are calculated with  $\frac{1}{2T'} = 5.62$ ,  $\beta' = 0.13 \cdot 10^{-1}$  and  $\frac{1}{2T} = 5.75$ ,  $a = -0.78 \cdot 10^{-3}$ ,  $b = 0.92 \cdot 10^{-2}$  for the  $q$ - and  $qp$ -rotor models, respectively. The experimental values  $(Q_0)_{\text{ex}}$  as well as the upper and lower experimental errors  $\Delta Q_0^+$  and  $\Delta Q_0^-$  are taken from Ref. 61.

Table 9. Intrinsic electric quadrupole moments for  $^{196}\text{Pb}$  in units of eb. The theoretical moments  $(Q_0)_q$  and  $(Q_0)_{qp}$  are calculated with  $\frac{1}{2T'} = 5.68$ ,  $\beta' = 0.11 \cdot 10^{-1}$  and  $\frac{1}{2T} = 5.71$ ,  $a = -0.17 \cdot 10^{-3}$ ,  $b = 0.11 \cdot 10^{-1}$  for the  $q$ - and  $qp$ -rotor models, respectively. The experimental values  $(Q_0)_{\text{ex}}$  as well as the upper and lower experimental errors  $\Delta Q_0^+$  and  $\Delta Q_0^-$  are taken from Ref. 57.

**Figure captions:**

Fig. 1. The characterization, in the plane of the free parameters  $a = \beta \cos \gamma$  and  $b = \beta \sin \gamma$  of the  $qp$ -rotor model, of the SD bands in the  $A \sim 130, 150$  and  $190$  mass regions.

Fig. 2. Comparison between the theoretical and experimental  $\gamma$ -ray energies in keV. Solid lines and dotted lines display the results for the  $qp$ -rotor model and the basic model, respectively.

Fig. 3. Comparison between the theoretical and experimental  $\gamma$ -ray energies in keV. Solid lines and dotted lines display the results for the  $qp$ -rotor model and the  $q$ -rotor model, respectively.

Fig. 4. Comparison between the theoretical and experimental  $\gamma$ -ray energies in keV. Solid lines and dotted lines display the results for the  $qp$ -rotor model and the  $q$ -rotor model, respectively.

Fig. 5. Comparison between the theoretical and experimental  $\gamma$ -ray energies in keV. Solid lines and dotted lines display the results for the  $qp$ -rotor model and the  $q$ -rotor model, respectively.

Fig. 6. Comparison between the theoretical and experimental  $\gamma$ -ray energies in keV. Solid lines and dotted lines display the results for the  $qp$ -rotor model and the  $q$ -rotor model, respectively.

Fig. 7. Comparison between the theoretical and experimental  $\gamma$ -ray energies in keV. Solid lines and dotted lines display the results for the  $qp$ -rotor model and the  $q$ -rotor model, respectively.

Fig. 8. The dynamical moments of inertia for  $^{190-192}\text{Hg}$ ,  $^{194}\text{Hg(a)}$  and  $^{194}\text{Hg(b)}$  calculated for the  $U_q(\text{su}_2)$  and  $U_{qp}(\text{u}_2)$  symmetries and compared to the experimental values. The moments of inertia are in units of  $\hbar^2\text{MeV}^{-1}$ .

Fig. 9. The geometrical factor  $G(I)$  of the reduced transition probability  $B(E2)$  for  $^{192}\text{Hg}$ ,  $^{194}\text{Pb}$  and  $^{196}\text{Pb}$  calculated for the  $U_q(\text{su}_2)$ ,  $U_{qp}(\text{u}_2)$  and  $\text{su}_2$  symmetries.